TRANSFORMATION OF CARTESIAN TO GEODETIC COORDINATES WITHOUT ITERATIONS

By Rey-Jer You[1]

ABSTRACT: A noniterative transformation of earth-centered, earth-fixed (ECEF) Cartesian coordinates to geodetic coordinates of a point is presented. The transformation is based on ellipsoidal coordinates. The ECEF Cartesian coordinates of a point are first transformed to the ellipsoidal coordinates by closed formulas. Second, the reduced latitude referred to the reference ellipsoid takes the approximation to the reduced latitude referred to a confocal ellipsoid. Third, the approximation can be improved by a correction for the purpose of higher accuracy. The accuracy of the transformation is analyzed in the paper and it is shown that the derived geodetic coordinates are sufficiently accurate for the most geodetic purposes.

INTRODUCTION

The transformation between the Cartesian and geodetic coordinates is performed more frequently since the popularization of satellite positioning, such as positioning from the global positioning system (GPS). From satellite po-sitioning one normally obtains the three-dimensional ECEF Cartesian coor-dinates of points. To be desired is the ability to find the geodetic coordinates for geodetic applications, such as investigating the earth's gravity field, map projections, etc. There have been numerous sources working toward this transformation. To transform the Cartesian coordinates to the geodetic coor-dinates, two methods are usually found in literature. One of them proposes the use of approximations. Some authors have worked with an iterative pro-cess for the approximation (e.g., Heiskanen and Moritz 1967; Vicenty 1978; Heiskanen 1982; Benning 1987; Heck 1987). Other approximations are based on a noniterative process; for example Vincenty (1980), who brought in an auxiliary ellipsoid to solve the transformation problem and obtained relatively accurate results.

In the second method, closed formulas are put forward for consideration. Several closed forms are based on solutions of a fourth-order equation or a cubic equation of an angle variable, such as geodetic latitude, reduced latitude referred to the reference ellipsoid (e.g., Penev 1978; Ozone 1985; Vanicek and Krakiwsky 1986; Borkowski 1987, 1989; Lapaine 1990). Grafarend and Lohse (1991) developed their closed form using the principle of minimal distance mapping. Another closed form is based on the work of Bowring (Bowring 1976, 1985; Hofmann-Wellenhof et al. 1992; Soler et al. 1992). They introduced the parametric latitude of the foot point of the spatial point as an auxiliary angle variable. Bowring and Hofmann-Wellenhof formulas are identical and work quite well for small values of ellipsoidal height. The Soler method gives excellent results for very large values of ellipsoidal height. Here, the ECEF Cartesian coordinates are converted into the geodetic coordinates by means of the ellipsoidal coordinates introduced in connection with the normal gravity by Somigliana (e.g., Heiskanen and Moritz 1967).

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[1]Assoc. Prof., Dept. of Surv. Engrg., National Cheng Kung Univ., 1 University Rd., Tainan, Taiwan, ROC.

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Using this concept makes the method better and easier to understand. The algorithm is short, simple, and the solution is accurate enough for practical use.

TRANSFORMATION BY MEANS OF ELLIPSOIDAL COORDINATES

The relationship between the ECEF Cartesian coordinates (x, y, z) and the geodetic coordinates (ϕ, λ, h) of a point is given by

$$x = (N + h)\cos\phi\cos\lambda; \quad y = (N + h)\cos\phi\sin\lambda \tag{1a,b}$$

$$z = \left(\frac{b^2}{a^2}N + h\right)\sin\phi \tag{1c}$$

where ϕ, λ, and h represent the geodetic latitude, geodetic longitude, and height above the reference ellipsoid, respectively. The major and minor semi-axes of the reference ellipsoid are given by a and b, and N is the radius of curvature in the prime vertical:

$$N(\phi) = \frac{a^2}{(a^2\cos^2\phi + b^2\sin^2\phi)^{1/2}} \tag{2}$$

If the geodetic coordinates of a point P are known, it is easy to obtain Cartesian coordinates by the formulas in (1). But if the spatial Cartesian coordinates are given, for instance from GPS positioning, it is more difficult to find the corresponding geodetic coordinates, especially the solution of ϕ and h, since (1c) involves the quantity a^2/b^2. In the following, a simple al-gorithm for the inverse transformation problem with the aid of the ellipsoidal coordinates is given.

It is assumed that an ellipsoid of revolution passes through a point P in space, and the point R is the foot point on the reference ellipsoid. This ellip-soid and the reference ellipsoid have the same center and linear eccentricity $E = \sqrt{a^2 - b^2}$, i.e., both ellipsoids are confocal. The axes of both ellipsoids are coincident with each other (Fig. 1). Point P can be specified by the ellipsoidal coordinates: semiminor axis u, reduced latitude β_0, and geodetic longitude. The relation of the Cartesian and ellipsoidal coordinates is given as follows (e.g., Torge 1991):

$$x = \sqrt{u^2 + E^2}\cos\beta_0\cos\lambda; \qquad y = \sqrt{u^2 + E^2}\cos\beta_0\sin\lambda; \qquad z = u\sin\beta_0 \tag{3a–c}$$

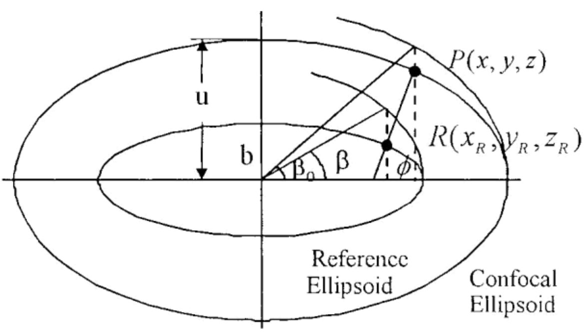

FIG. 1. Reference Ellipsoid and Confocal Ellipsoid

The parameter $\sqrt{u^2 + E^2}$ is the semimajor axis of the confocal ellipsoid. Let $u = b$, then the ellipsoid is the reference ellipsoid itself. It is not difficult to find the inverse transformation for $u(x, y, z)$, $\beta_0(x, y, z)$, $\lambda(x, y, z)$. The results are listed in the following:

$$u = \sqrt{\frac{1}{2}(r^2 - E^2) + \frac{1}{2}\sqrt{(r^2 - E^2)^2 + 4E^2 z^2}} \tag{4a}$$

$$\tan\beta_0 = \frac{\sqrt{u^2 + E^2}}{u} \frac{z}{\sqrt{x^2 + y^2}}; \quad \tan\lambda = \frac{y}{x} \tag{4b,c}$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

It can be easily proved that β and β_0 are not the same, if $u \neq b$. Never-theless, the basic idea is to take the reduced latitude β_0 of the point P on the confocal ellipsoid as the reduced latitude β of the point R on the reference ellipsoid, i.e.

$$\tan\beta_{(0)} \approx \tan\beta_0 \tag{5}$$

where the index (0) denotes the approximation of zero order. The geodetic latitude ϕ of the point R or point P referred to the reference ellipsoid is given by the well-known formula

$$\tan\phi = \frac{a}{b}\tan\beta \tag{6}$$

and the ellipsoidal height h by

$$h = \sqrt{(z - b\sin\beta)^2 + (Q - a\cos\beta)^2} \tag{7}$$

where $Q = \sqrt{x^2 + y^2}$. Once the reduced latitude of zero-order approximation is given, the ellipsoidal height of zero-order approximation can be obtained by substituting β_0 for β in (7).

To discuss the error of the zero-order approximation in detail, begin with the colinearity of points P and R:

$$z = z_R + (Q - q)\tan\phi \tag{8}$$

where $q = \sqrt{x_R^2 + y_R^2}$. The points P and R lie, respectively, on the confocal ellipsoid and the reference ellipsoid. They must satisfy

$$Q = \sqrt{u^2 + E^2}\cos\beta_0; \quad z = u\sin\beta_0 \tag{9}$$

and

$$q = a\cos\beta; \quad z_R = b\sin\beta \tag{10}$$

which lead to the desired equation

$$bu\sin\beta_0 = a\sqrt{u^2 + E^2}\cos\beta_0\tan\beta - E^2\sin\beta \tag{11}$$

for the error analysis. Let $\beta = \beta_0 + \Delta\beta$, and the right-hand side of (11) is expanded by the Taylor series at $\beta = \beta_0$:

$$bu\sin\beta_0 = a\sqrt{u^2 + E^2}\cos\beta_0\tan(\beta_0 + \Delta\beta) - E^2\sin(\beta_0 + \Delta\beta)$$

$$= (a\sqrt{u^2 + E^2} - E^2)\sin\beta_0 + (a\sqrt{u^2 + E^2}\sec\beta_0 - E^2\cos\beta_0)\Delta\beta + \cdots \tag{12}$$

Consequently, the error in β is approximate to

$$\Delta\beta = f(u, \beta_0) \approx \frac{(bu \_ a\sqrt{u^2 \_ E^2} \_ E^2)\sin\beta_0}{(a\sqrt{u^2 \_ E^2}\sec\beta_0 \_ E^2\cos\beta_0)} \qquad (13)$$

Setting the first derivative of the function f equal to zero —

$$f'(\beta_0) = \frac{(bu \_ a\sqrt{u^2 \_ E^2} \_ E^2)(a\sqrt{u^2 \_ E^2} \_ a\sqrt{u^2 \_ E^2}\tan\beta_0 \_ E^2)}{(a\sqrt{u^2 \_ E^2}\sec\beta_0 \_ E^2\cos\beta_0)^2} \qquad (14)$$

— the maximal error $|\Delta\beta|_{max}$ under $u = const.$ can then be found in

$$\beta_0 = \arctan\sqrt{1 \_ \frac{E^2}{a\sqrt{u^2 \_ E^2}}} \approx 45^\circ \qquad (15)$$

Next, the function $h(\beta)$ in (7) is also expanded by the Taylor series:

$$h(\beta) = \sqrt{(z \_ b\sin(\beta_0 \_ \Delta\beta))^2 \_ (Q \_ a\cos(\beta_0 \_ \Delta\beta))^2}$$

$$= h(\beta_0) \_ h'(\beta_0) \_ \Delta\beta \_ \frac{1}{2} h''(\beta_0) \_ \Delta\beta^2 \_ \_ \_ \_ = h(\beta_0) \_ \Delta h \qquad (16)$$

Because the value of $h''(\beta_0) \_ \Delta\beta^2/2$ and the value of $h'(\beta_0) \_ \Delta\beta$ are the same order, it is necessary to estimate more carefully the corresponding error of ellipsoidal height, and the second derivative may be taken into consideration. Hence, the error of ellipsoidal height is approximately equal to

$$\Delta h = h_{should} \_ h_{computed} = \Delta\beta(a\sqrt{u^2 \_ E^2} \_ E^2 \_ bu)\cos\beta_0 \sin\beta_0 / T^{1/2}$$

$$\_ \Delta\beta^2(2(a\sqrt{u^2 \_ E^2} \_ bu \_ (a\sqrt{u^2 \_ E^2} \_ bu \_ 2E^2)\cos 2\beta_0)T$$

$$\_ (E^2 \_ a\sqrt{u^2 \_ E^2} \_ bu)^2 \sin^2 2\beta_0)/(8T^{3/2}) \qquad (17)$$

where $T = (a \_ \sqrt{u^2 \_ E^2})^2 \cos^2\beta_0 \_ (b \_ u)^2 \sin^2\beta_0$. Figs. 2 and 3 illustrate the error estimation of the zero-order approximation in reduced latitude and in the ellipsoidal height with respect to the reference ellipsoid.

The accuracy of the reduced latitude referred to the reference ellipsoid can be improved by adding the correction (13) to $\beta_{(0)}$. In this case, one obtains

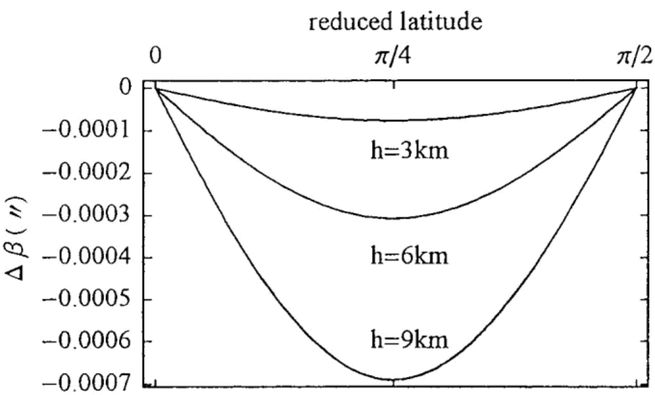

FIG. 2. Error of Reduced Latitude of Zero Order

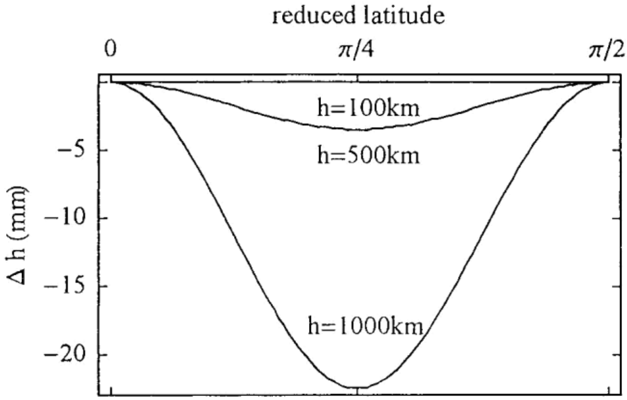

FIG. 3. Error of Ellipsoidal Height of Zero Order

the approximation of first order. The ellipsoidal height of first-order approximation can be obtained by substituting $\beta_{(1)}$ for β in (7).

NUMERICAL EXAMPLES

The transformation results of some numerical examples by means of the noniterative approximation of zero and first order as well as the iterative method

$$\varphi_{(i+1)} = \arctan\left[\frac{z}{\sqrt{x^2+y^2}}\left(1-\frac{e^2N(\varphi_{(i)})}{N(\varphi_{(i)})+h_{(i)}}\right)^{-1}\right] \qquad (18a)$$

$$h_{(i)} = \frac{\sqrt{x^2+y^2}}{\cos\varphi_{(i)}} - N(\varphi_{(i)}) \qquad (18b)$$

are given in Table 1. In (18), e = first eccentricity of the reference ellipsoid. The iterative method can be found in many geodesy books (e.g., Heck 1987). Eq. (18b) is numerically stable, if the value of φ is less than 45°. The stability problem is not of concern in this paper. The reader can refer to Wolf et al. (1997) for more detail.

Because of their symmetry, only the cases for $0° \le \varphi \le 90°$ are discussed in the following. The convergence conditions of the iterative method are 0.0001″ in latitude and 1 mm in height. The number in the parentheses in the last column denotes the number of iterations. It is shown that there are almost no differences between the iterative approximation and noniterative approximation of zero order, if the ellipsoidal height is within 3 km. There-fore, the accuracy of the approximation of zero order is sufficient for most geodetic and navigation purposes. If the ellipsoidal height is 100 km over the reference ellipsoid (Case 7), one can achieve the maximum difference in latitude 0.08″ and in height 0.2 mm. Even for the last case in Table 1, the maximum difference in latitude is only 6.38″ and in height 2.3 cm. For the purposes of higher accuracy, one can use the first-order approximation. In Table 1, one can see that there are no differences between the first order and iterative methods, if the ellipsoidal height does not exceed 800 km. For the case h = 1,000 km, the difference in latitude is only 0.0002″, with no difference in height.

TABLE 1. Numerical Examples

Case (1)	Cartesian coordinates (2)	Noniterative Method: Zero order (3)	Noniterative Method: First order (4)	Iterative method (5)
1	*x* = _2,259,148.993	f = 45700900.00000	f = 45700900.00000	f = 45700900.00000
	y = 3,912,960.837	l = 120700900.00000	l = 120700900.00000	l = 120700900.00000
	z = 4,488,055.516	*h* = 1,000.000 m	*h* = 1,000.000 m	*h* = 1,000.000 m (4)
2	*x* = _2,259,502.546	f = 45700900.00000	f = 45700900.00000	f = 45700900.00000
	y = 3,913,573.210	l = 120700900.00000	l = 120700900.00000	l = 120700900.00000
	z = 4,488,762.622	*h* = 2,000.000 m	*h* = 2,000.000 m	*h* = 2,000.000 m (4)
3	*x* = _2,259,856.100	f = 45700900.00010	f = 45700900.00000	f = 45700900.00000
	y = 3,914,185.582	l = 120700900.00000	l = 120700900.00000	l = 120700900.00000
	z = 4,489,469.729	*h* = 3,000.000 m	*h* = 3,000.000 m	*h* = 3,000.000 m (5)
4	*x* = _2,260,209.653	f = 45700900.00010	f = 45700900.00000	f = 45700900.00000
	y = 3,914,797.955	l = 120700900.00000	l = 120700900.00000	l = 120700900.00000
	z = 4,490,l76.836	*h* = 4,000.000 m	*h* = 4,000.000 m	*h* = 4,000.000 m (5)
5	*x* = _2,262,330.973	f = 45700900.00090	f = 45700900.00000	f = 45700900.00000
	y = 3,918,472.189	l = 120700900.00000	l = 120700900.00000	l = 120700900.00000
	z = 4,494,419.477	*h* = 10,000.000 m	*h* = 10,000.000 m	*h* = 10,000.000 m (5)
6	*x* = _2,265,866.507	f = 45700900.00340	f = 45700900.00000	f = 45700900.00000
	y = 3,924,595.914	l = 120700900.00000	l = 120700900.00000	l = 120700900.00000
	z = 4,501,490.544	*h* = 20,000.000 m	*h* = 20,000.000 m	*h* = 20,000.000 m (5)
7	*x* = _2,294,150.778	f = 45700900.08280	f = 45700900.00000	f = 45700900.00000
	y = 3,973,585.709	l = 120700900.00000	l = 120700900.00000	l = 120700900.00000
	z = 4,558,059.087	*h* = 100,000.000 m	*h* = 100,000.000 m	*h* = 100,000.000 m (5)
8	*x* = _2,541,638.152	f = 45700904.31510	f = 45700900.00010	f = 45700900.00000
	y = 4,402,246.414	l = 120700900.00000	l = 120700900.00000	l = 120700900.00000
	z = 5,053,033.834	*h* = 800,000.013 m	*h* = 800,000.000 m	*h* = 800,000.001 m (5)
9	*x* = _2,612,348.830	f = 45700906.35120	f = 45700900.00020	f = 45700900.00000
	y = 4,524,720.901	l = 120700900.00000	l = 120700900.00000	l = 120700900.00000
	z = 5,194,455.190	*h* = 1,000,000.023 m	*h* = 1,000,000.000 m	*h* = 1,000,000.000 m (5)

Note: Geodetic coordinates are related to WGS84.

CONCLUSIONS

A noniterative transformation of the ECEF Cartesian coordinates to geo-detic coordinates based on the ellipsoidal coordinates has been developed. The results are most satisfactory. For a point whose height is within 3 km over the reference ellipsoid, the approximation of zero order is accurate enough. The approximation of zero order is already very suitable for the most geodetic applications, since the regions of the most geodetic activities are seldom over 3 km high. For the regions with higher altitude and precise navigation, the approximation of first order can be used in order to obtain the higher accuracy. If a user programs or makes the first-order solution ''standard,'' one never need worry whether an answer is ''good enough.''

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